A Synopsis On the Forward, Discount, and Zero Interest Rate Functions

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1 The Three Curves Describing the Three Functions

The identities begin with having the forward rate function, $\boldsymbol{FR}$, which generates the discount function $\boldsymbol{DF}$. The discount function tells you how much \$1 paid at future year T is worth today. Note that the zero curve, $\boldsymbol{ZR}$, is just the average (dividing by T) of the area under the forward rate curve, $\boldsymbol{FR}$, between 0 and T.

$$\begin{aligned} \boldsymbol{DF(T)} &= \boldsymbol{e}^{-\int_0^T \boldsymbol{FR(t)}\,\boldsymbol{dt}} \\ \boldsymbol{ZR(T)} &= \tfrac{\boldsymbol{1}}{\boldsymbol{T}} \int_{\boldsymbol{0}}^{\boldsymbol{T}} \boldsymbol{FR(t)dt}, \text{ so } \boldsymbol{e}^{-\boldsymbol{T}\times \boldsymbol{ZR(T)}} = \boldsymbol{DF(T)} \\ \boldsymbol{FR(T)} &= \boldsymbol{T}\,\tfrac{\boldsymbol{\partial ZR(t)}}{\boldsymbol{\partial t}}\,|_{\boldsymbol{t=T}} + \boldsymbol{ZR(T)} \end{aligned} \tag{1}$$

The third equation gives a useful comparison between the FR and ZR curves: whenever ZR is increasing, FR lies above ZR while whenever ZR is decreasing, FR lies below ZR.

For an additional geometrical interpretation, suppose we select $T = 16$ in Figure 1. Because the vertical axis is in %, it follows that the intersection of the vertical line at 16 with the zero curve is the decimal 0.035 annual zero rate. This means that

$$\boldsymbol{DF = e^{-16\times 0.035} = \$0.5712},$$

which is what \$1 paid at year 16, with no other paymants, is worth today.

1.1 Some Central Bank Methods for Computing the Zeros

Table 1: Forward Rate Curve Fitting Procedures of 12 Central Banks

Central Bank	Curve Fitting Procedure
Belgium	Nelson–Siegel, Svensson
Canada	Svensson
Finland	Nelson–Siegel
France	Nelson–Siegel, Svensson
Germany	Nelson–Siegel, Svensson
Italy	Nelson–Siegel
Japan	Smoothing Splines
Norway	Svensson
Spain	Nelson–Siegel(before 1995), Svensson
Sweden	Svensson
UK	Svensson
USA	Smoothing Splines

1.2 Illustrating the Svensson Forward Rate function [17]

Here t represents time in years, while other notations are for the 5 parameters. Among the users are: David Bolder(Bank of Canada) and Handy Yunianto(Mandira Securities, Jarakta, Indonesia).

$$\boldsymbol{FR}(\boldsymbol{t} \mid \boldsymbol{\beta_0}, \boldsymbol{\beta_1}, \boldsymbol{\beta_2}, \boldsymbol{\beta_3}, \boldsymbol{\tau_1}, \boldsymbol{\tau_2}) =$$

$$\boldsymbol{\beta_0} + \boldsymbol{\beta_1}\, e^{-t/\tau_1} + \boldsymbol{\beta_2}\, (t/\tau_1)\, e^{-t/\tau_1} + \boldsymbol{\beta_3}\, (t/\tau_2)\, e^{-t/\tau_2}$$

2 A FR Extraction from WSJ Treasury Quotes for 03/20/2014

A common accuracy measure we use is given in the following definition, where using an optimization routine we compute present values (prices) of all instruments and would like these to be close to the WSJ published prices. In this regard the smaller MAPE is, the better.

Definition 2.1 *For an observed bond price time series* $P_t,\ t = \overline{1,n}$ *and a computed price series,* $\hat{P}_t$*, the mean absolute percentage error, MAPE, is*

$$MAPE = \sum_{t=1}^{n} \frac{\left|P_t - \hat{P}_t\right|}{P_t} \times \frac{100}{n}.$$

The units are percentages, so 0.05 means 5 basis points, where one basis point is 0.01%.

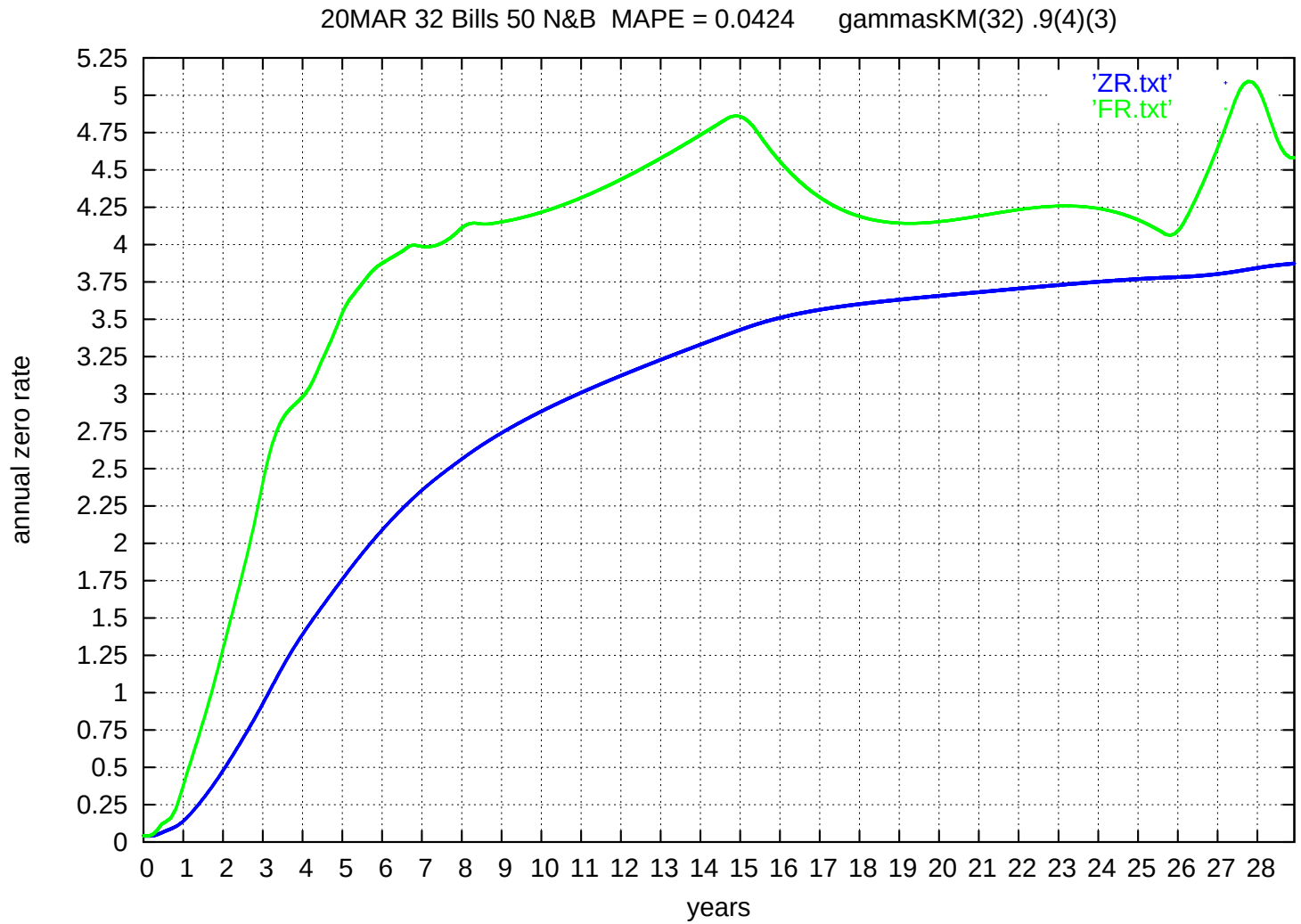

Figure 1: A Very Recent FR Extraction

3 Remarks on the Modus Operandi

The Preface of the Kortanek–Medvedev book [5, xi-xii] gives a description on the methodology we are following in a treatment of uncertainty. A shorter and more elegant description appear in Gusev and Romanov book [4, 299-300], which we quote:

> "A conventional approach to the study of uncertain systems relates to the assumption that uncertainty may be described as a random process with known characteristics. In many applied problems, however, there may be a limited number of observations, incomplete knowledge of the data, and no available statistics whatever. An alternative approach to the uncertainty treatment, known as guaranteed (see references), is based on set–membership (unknown but bounded) error description. In the problems considered here the set–membership of uncertainty is employed."

Other research employing the set–membership of uncertainty include: [11], [1], [8], [9], [2], [10], [3], [18], [12].

Major contributions in computer implementation of the set–membershiop approach to uncertainty occurred by the second author; [13, 14, 15, 16].

4 Daily Living with the Procedure

Every morning I could consider the (about) 277 notes and bonds and 32 bills appearing in U.S. Treasury quotes and extract a zero curve and a forward rate curve. I keep all 32 bills. Notes or Bonds whose maturities are within those of the Bills maturities are deleted because the bills having no coupons, usually about 32 or so, already supply the initial, low maturity, section of the zero rate curve which is intimately related to the FR, see the third equation in (1). We denote the Notes & Bonds remaining as the *NON-Early bonds*.

I needed an objective way to eliminate some of the Notes and Bonds, hoping that what remains will be representative of the total. What I do is objective, but could be considered "ad hoc". I specify a bp filter value, say 0.07 down to 0.01. For those Notes or Bonds whose published ask-yield is not within the specified bp filter, I ignore them. I was motivated to do this from the obvious case bonds which are callable or inflation protected. It is very gratifying that the Newton-Raphson IRR is very often "close" to the published ask-yields (as they should be.) I can run the full gamut of 277 notes and bonds but must reduce the accuracy parameters to 0.9, see the next paragraph.

For the Figure 1 I used 0.016 which resulted in 50 admissible Notes or Bonds. I then specify an accuracy target, say 0.9999 for all Bills and 0.999 for all the Notes and Bonds, which represent the tightness of computed present values to published market prices, see [6, (32)] .

I have presented this appoach at seminars and conferences (most recent being Quantitative Methods in Finance 2014 in Sydney, Australia, where I stressed an application to Pension Planning), and some financial institutions, such as the Macquarie Bank and Westpac in Australia at the Intelligence Finance Conference in Melbourne, 13–14 December, 2004: "Nonstochastic Uncertainty Approaches to Prediction in Finance", pages144–171, [7].

I never took the position of stating that our procedure should replace what is being used; rather it is another" mirror" reflecting an extracted FR curve, noting that the problem is essentially unsolvable.

Finally, Handy Yunianto of Mandiri Securities in Jakarta collaborated with me on our methodologu and met me in Rome for a conference in June, 2004.

5 Presenting Detail on the WSJ Treasury Quotations having non–early Notes & Bonds

The Treasury Quotes are available as HTML filed from the internet version of the Wall Street Journal, but for completeness I present the data we used for the 03/20/2015 FR extraction appearing in Figure 1. From the very outset we delete those bonds whose maturities are within those of all the Bills because the Treasury Bill data are already "classified" as zero-coupon bonds having the shortest maturities available.

5.1 All Bills and the NON–Early Notes & Bonds in the 03/20/2014 Data Set

The original number of Notes & Bonds is 280, but the non-earlies number 241. Together with the original 32 Bills we have 273 instruments.

Table 2: T-Bills 03/20/2014

T-Bill	01	Maturity	03/27/2014	YR =	4.021918	AskYield	0.000410	Price	99.835237
T-Bill	02	Maturity	04/03/2014	YR =	4.041096	AskYield	0.000300	Price	99.878841
T-Bill	03	Maturity	04/10/2014	YR =	4.060274	AskYield	0.000460	Price	99.813402
T-Bill	04	Maturity	04/17/2014	YR =	4.079452	AskYield	0.000560	Price	99.771811
T-Bill	05	Maturity	04/24/2014	YR =	4.098630	AskYield	0.000560	Price	99.770740
T-Bill	06	Maturity	05/01/2014	YR =	4.117808	AskYield	0.000460	Price	99.810760
T-Bill	07	Maturity	05/08/2014	YR =	4.136986	AskYield	0.000460	Price	99.809880
T-Bill	08	Maturity	05/15/2014	YR =	4.156164	AskYield	0.000350	Price	99.854640
T-Bill	09	Maturity	05/22/2014	YR =	4.175342	AskYield	0.000350	Price	99.853970
T-Bill	10	Maturity	05/29/2014	YR =	4.194521	AskYield	0.000410	Price	99.828172
T-Bill	11	Maturity	06/05/2014	YR =	4.213699	AskYield	0.000460	Price	99.806358
T-Bill	12	Maturity	06/12/2014	YR =	4.232877	AskYield	0.000460	Price	99.805477
T-Bill	13	Maturity	06/19/2014	YR =	4.252055	AskYield	0.000560	Price	99.762168
T-Bill	14	Maturity	06/26/2014	YR =	4.271233	AskYield	0.000460	Price	99.803716
T-Bill	15	Maturity	07/03/2014	YR =	4.290411	AskYield	0.000460	Price	99.802836
T-Bill	16	Maturity	07/10/2014	YR =	4.309589	AskYield	0.000460	Price	99.801955
T-Bill	17	Maturity	07/17/2014	YR =	4.328767	AskYield	0.000560	Price	99.757883
T-Bill	18	Maturity	07/24/2014	YR =	4.347945	AskYield	0.000560	Price	99.756811
T-Bill	19	Maturity	07/31/2014	YR =	4.367123	AskYield	0.000350	Price	99.847267
T-Bill	20	Maturity	08/07/2014	YR =	4.386301	AskYield	0.000560	Price	99.754669
T-Bill	21	Maturity	08/14/2014	YR =	4.405479	AskYield	0.000560	Price	99.753597
T-Bill	22	Maturity	08/21/2014	YR =	4.424658	AskYield	0.000610	Price	99.730460
T-Bill	23	Maturity	08/28/2014	YR =	4.443836	AskYield	0.000560	Price	99.751455
T-Bill	24	Maturity	09/04/2014	YR =	4.463014	AskYield	0.000660	Price	99.705874
T-Bill	25	Maturity	09/11/2014	YR =	4.482192	AskYield	0.000710	Price	99.682270
T-Bill	26	Maturity	09/18/2014	YR =	4.501370	AskYield	0.000760	Price	99.658480
T-Bill	27	Maturity	10/16/2014	YR =	4.578082	AskYield	0.000710	Price	99.675484
T-Bill	28	Maturity	11/13/2014	YR =	4.654795	AskYield	0.000910	Price	99.577310
T-Bill	29	Maturity	12/11/2014	YR =	4.731507	AskYield	0.000960	Price	99.546805
T-Bill	30	Maturity	01/08/2015	YR =	4.808219	AskYield	0.001010	Price	99.515547

Table 3: T-Bills 03/20/2014

T-Bill	31	Maturity	02/05/2015	YR =	4.884932	AskYield	0.001120	Price	99.454382
T-Bill	32	Maturity	03/05/2015	YR =	4.961644	AskYield	0.001270	Price	99.371852

Table 4: Bonds and Notes 03/20/2014

Maturity	Coupon	Bid	Asked	Chg	AskYield
03/15/2015	0.375	100.2070	100.2305	-0.0039	0.140
03/31/2015	0.250	100.0938	100.1016	unch.	0.151
03/31/2015	2.500	102.3984	102.4063	-0.0078	0.155
04/15/2015	0.375	100.2188	100.2266	unch.	0.163
04/30/2015	0.125	099.9297	099.9766	-0.0078	0.146
04/30/2015	2.500	102.5781	102.5859	unch.	0.168
05/15/2015	0.250	100.0742	100.0977	-0.0117	0.165
05/15/2015	4.125	104.5469	104.5547	-0.0078	0.165
05/31/2015	0.250	100.0547	100.1016	-0.0078	0.165
05/31/2015	2.125	102.2969	102.3125	-0.0078	0.187
06/15/2015	0.375	100.2188	100.2344	-0.0078	0.185
06/30/2015	0.375	100.2188	100.2266	-0.0078	0.198
06/30/2015	1.875	102.1250	102.1328	-0.0234	0.204
07/15/2015	0.250	100.0313	100.0391	-0.0156	0.220
07/31/2015	0.250	100.0078	100.0469	-0.0156	0.216
07/31/2015	1.750	102.0625	102.0938	unch.	0.213
08/15/2015	0.250	100.0078	100.0547	0.0313	0.211
08/15/2015	4.250	105.5781	105.6016	-0.1328	0.256
08/15/2015	10.625	114.5703	114.5938	0.0078	0.224
08/31/2015	0.375	100.1641	100.1797	0.0234	0.250
08/31/2015	1.250	101.4375	101.4453	0.0078	0.246
09/15/2015	0.250	099.9609	099.9844	0.0391	0.261
09/30/2015	0.250	099.9297	099.9688	0.0078	0.271
09/30/2015	1.250	101.4766	101.4844	0.0078	0.275
10/15/2015	0.250	099.9141	099.9375	0.0234	0.290
10/31/2015	0.250	099.8828	099.9219	-0.0078	0.299
10/31/2015	1.250	101.4766	101.5000	-0.0234	0.315
11/15/2015	0.375	100.0625	100.0859	0.0156	0.323
11/15/2015	4.500	106.8438	106.8672	-0.0391	0.328
11/15/2015	9.875	115.6875	115.7109	-0.0391	0.330

Table 5: Notes and Bonds 03/20/2014

11/30/2015	0.250	099.8438	099.8750	unch.	0.324
11/30/2015	1.375	101.7422	101.7656	0.0156	0.330
12/15/2015	0.250	099.8125	099.8203	0.0156	0.354
12/31/2015	0.250	099.7891	099.8281	0.0156	0.347
12/31/2015	2.125	103.0938	103.1016	-0.0078	0.374
01/15/2016	0.375	099.9688	099.9922	0.0156	0.379
01/31/2016	0.375	099.9375	099.9766	unch.	0.388
01/31/2016	2.000	102.9688	102.9766	-0.0156	0.396
02/15/2016	0.375	099.9219	099.9297	0.0234	0.412
02/15/2016	4.500	107.7500	107.7734	-0.0313	0.402
02/15/2016	9.250	116.8203	116.8281	0.0234	0.381
02/29/2016	0.250	099.6328	099.6719	-0.0156	0.420
02/29/2016	2.125	103.2813	103.2891	-0.0078	0.423
02/29/2016	2.625	104.2344	104.2656	-0.0156	0.418
03/15/2016	0.375	099.8477	099.8711	0.0195	0.440
03/31/2016	2.250	103.5859	103.6172	-0.0313	0.456
03/31/2016	2.375	103.8672	103.8828	-0.0078	0.449
04/15/2016	0.250	099.4922	099.5313	-0.0078	0.478
04/30/2016	2.000	103.1328	103.1641	-0.0234	0.491
04/30/2016	2.625	104.4531	104.4844	-0.0234	0.487
05/15/2016	0.250	099.3984	099.4375	-0.0078	0.513
05/15/2016	5.125	109.8672	109.8828	-0.0625	0.502
05/15/2016	7.250	114.3906	114.4141	-0.0156	0.506
05/31/2016	1.750	102.6250	102.6406	-0.0313	0.538
05/31/2016	3.250	105.8906	105.9297	-0.0313	0.529
06/30/2016	1.500	102.0781	102.0938	0.0078	0.574
06/30/2016	3.250	106.0469	106.0703	-0.0625	0.565
07/15/2016	0.625	100.0391	100.0781	unch.	0.591
07/31/2016	1.500	102.0859	102.1016	0.0078	0.603
07/31/2016	3.250	106.2109	106.2344	-0.0078	0.591

Table 6: Notes and Bonds 03/20/2014

08/15/2016	0.625	099.9531	099.9922	-0.0234	0.628
08/15/2016	4.875	110.1484	110.1953	-0.0234	0.600
08/31/2016	1.000	100.8281	100.8594	-0.0078	0.645
08/31/2016	3.000	105.7109	105.7344	-0.0078	0.631
09/15/2016	0.875	100.4766	100.5156	-0.0313	0.665
09/30/2016	1.000	100.7734	100.7813	unch.	0.688
09/30/2016	3.000	105.7891	105.8203	-0.0469	0.674
10/15/2016	0.625	099.7656	099.8047	0.0078	0.702
10/31/2016	1.000	100.6953	100.7109	-0.0313	0.725
10/31/2016	3.125	106.2109	106.2578	unch.	0.701
11/15/2016	0.625	099.6641	099.7031	-0.0078	0.738
11/15/2016	4.625	110.1641	110.2031	-0.0781	0.733
11/15/2016	7.500	117.7500	117.7734	-0.1094	0.721
11/30/2016	0.875	100.2891	100.2969	-0.0234	0.763
11/30/2016	2.750	105.2656	105.2891	-0.0313	0.763
12/15/2016	0.625	099.5469	099.5859	-0.0469	0.778
12/31/2016	0.875	100.1797	100.1953	-0.0469	0.804
12/31/2016	3.250	106.6953	106.7031	-0.0469	0.806
01/15/2017	0.750	099.7813	099.8203	-0.0391	0.815
01/31/2017	0.875	100.0781	100.0859	-0.0156	0.845
01/31/2017	3.125	106.4453	106.4688	-0.0547	0.835
02/15/2017	0.625	099.3047	099.3438	-0.0391	0.854
02/15/2017	4.625	110.8281	110.8750	-0.0234	0.830
02/28/2017	0.875	099.9609	099.9688	-0.0469	0.886
02/28/2017	3.000	106.1172	106.1641	-0.0625	0.874
03/15/2017	0.750	099.5547	099.5938	-0.0547	0.888
03/31/2017	1.000	100.2109	100.2266	-0.0469	0.924
03/31/2017	3.250	106.9766	107.0156	-0.0469	0.896
04/30/2017	0.875	099.7266	099.7344	-0.0547	0.962
04/30/2017	3.125	106.6094	106.6406	-0.0547	0.953

Table 7: Notes and Bonds 03/20/2014

05/15/2017	4.500	110.9375	110.9688	-0.0313	0.958
05/15/2017	8.750	124.1328	124.1641	-0.1172	0.949
05/31/2017	0.625	098.7969	098.8047	-0.0469	1.006
05/31/2017	2.750	105.4609	105.4922	-0.0391	0.999
06/30/2017	0.750	099.0234	099.0547	-0.0703	1.044
06/30/2017	2.500	104.6875	104.7188	-0.0391	1.032
07/31/2017	0.500	098.0391	098.0469	-0.0547	1.093
07/31/2017	2.375	104.2422	104.2734	-0.0469	1.078
08/15/2017	4.750	112.2422	112.2734	-0.0234	1.071
08/15/2017	8.875	126.0156	126.0469	-0.1563	1.067
08/31/2017	0.625	098.2578	098.2891	-0.0703	1.133
08/31/2017	1.875	102.5156	102.5469	-0.0469	1.119
09/30/2017	0.625	098.0859	098.1016	-0.0703	1.176
09/30/2017	1.875	102.4609	102.4922	-0.0391	1.152
10/31/2017	0.750	098.3594	098.3750	-0.0391	1.211
10/31/2017	1.875	102.3359	102.3672	-0.0781	1.203
11/15/2017	4.250	110.8750	110.9063	-0.0547	1.189
11/30/2017	0.625	097.7344	097.7500	-0.0547	1.250
11/30/2017	2.250	103.6563	103.6875	-0.0156	1.226
12/31/2017	0.750	098.0234	098.0313	-0.0391	1.285
12/31/2017	2.750	105.4688	105.5000	unch.	1.255
01/31/2018	0.875	098.3359	098.3438	-0.0469	1.316
01/31/2018	2.625	104.9453	104.9766	-0.0313	1.300
02/15/2018	3.500	108.2969	108.3281	-0.0313	1.306
02/28/2018	0.750	097.6641	097.6953	-0.0469	1.352
02/28/2018	2.750	105.3828	105.4141	-0.0625	1.336
03/31/2018	0.750	097.5078	097.5156	-0.0234	1.386
03/31/2018	2.875	105.8984	105.9297	-0.0078	1.357
04/30/2018	0.625	096.8281	096.8750	-0.0547	1.410
04/30/2018	2.625	104.8594	104.8906	-0.0391	1.396

Table 8: Notes and Bonds 03/20/2014

05/15/2018	3.875	109.9688	110.0000	-0.0547	1.388
05/15/2018	9.125	131.3203	131.3516	-0.1719	1.336
05/31/2018	1.000	098.1172	098.1641	-0.0625	1.453
05/31/2018	2.375	103.7969	103.8281	-0.0469	1.431
06/30/2018	1.375	099.5156	099.5469	-0.0703	1.485
06/30/2018	2.375	103.7969	103.8125	-0.0078	1.453
07/31/2018	1.375	099.3828	099.4297	-0.0547	1.510
07/31/2018	2.250	103.1641	103.2266	-0.0625	1.484
08/15/2018	4.000	110.7109	110.7266	0.0078	1.476
08/31/2018	1.500	099.7656	099.8125	-0.0703	1.544
09/30/2018	1.375	099.0781	099.1250	-0.0703	1.576
10/31/2018	1.250	098.3906	098.4375	-0.0859	1.603
10/31/2018	1.750	100.6953	100.7188	-0.0625	1.588
11/15/2018	3.750	109.6484	109.6797	-0.0703	1.583
11/15/2018	9.000	133.4063	133.4375	-0.1797	1.526
11/30/2018	1.250	098.2500	098.2578	-0.0625	1.637
11/30/2018	1.375	098.8750	098.8984	-0.0703	1.619
12/31/2018	1.375	098.6953	098.7188	-0.0625	1.655
12/31/2018	1.500	099.2344	099.2813	-0.0703	1.657
01/31/2019	1.250	097.9375	097.9453	-0.0391	1.692
01/31/2019	1.500	099.1016	099.1484	-0.0625	1.683
02/15/2019	2.750	104.9609	105.0234	-0.1094	1.679
02/15/2019	8.875	134.1328	134.1641	-0.1641	1.605
02/28/2019	1.375	098.3594	098.3828	-0.0781	1.718
02/28/2019	1.500	098.9844	099.0313	-0.0703	1.705
03/31/2019	1.500	098.8125	098.8203	-0.0781	1.746
04/30/2019	1.250	097.4375	097.4453	-0.0469	1.775
05/15/2019	3.125	106.6875	106.7109	-0.0703	1.757
05/31/2019	1.125	096.5703	096.6172	-0.0625	1.810
06/30/2019	1.000	095.7578	095.7891	-0.0469	1.841

Table 9: Notes and Bonds 03/20/2014

07/31/2019	0.875	094.8984	094.9453	-0.0625	1.870
08/15/2019	3.625	109.1797	109.1875	-0.0938	1.832
08/15/2019	8.125	132.5703	132.6172	-0.1094	1.771
08/31/2019	1.000	095.3359	095.3438	-0.0625	1.905
09/30/2019	1.000	095.1563	095.1719	-0.0625	1.925
10/31/2019	1.250	096.3047	096.3516	-0.0859	1.939
11/15/2019	3.375	107.8281	107.8750	-0.1172	1.899
11/30/2019	1.000	094.6797	094.7266	-0.1250	1.984
12/31/2019	1.125	095.2266	095.2734	-0.0547	1.995
01/31/2020	1.375	096.4297	096.4766	-0.0625	2.015
02/15/2020	3.625	109.1094	109.1563	-0.1250	1.975
02/15/2020	8.500	136.5938	136.6406	-0.1641	1.910
02/29/2020	1.250	095.5078	095.5547	-0.0781	2.048
03/31/2020	1.125	094.6250	094.6719	-0.0781	2.070
04/30/2020	1.125	094.4063	094.4531	-0.0938	2.097
05/15/2020	3.500	108.3359	108.3828	-0.1328	2.043
05/15/2020	8.750	139.0156	139.0625	-0.1016	1.975
05/31/2020	1.375	095.6875	095.7344	-0.1016	2.113
06/30/2020	1.875	098.4922	098.5156	-0.1094	2.129
07/31/2020	2.000	099.1484	099.1953	-0.1094	2.136
08/15/2020	2.625	102.7734	102.8203	-0.1250	2.151
08/15/2020	8.750	140.0078	140.0547	-0.0781	2.046
08/31/2020	2.125	099.6719	099.7188	-0.1094	2.172
09/30/2020	2.000	098.7891	098.8359	-0.0938	2.192
10/31/2020	1.750	097.1016	097.1484	-0.0547	2.216
11/15/2020	2.625	102.5859	102.6328	-0.1328	2.197
12/31/2020	2.375	100.7578	100.8047	-0.0859	2.246
01/31/2021	2.125	099.0156	099.0625	-0.0625	2.273
02/15/2021	3.625	108.7891	108.8359	-0.1406	2.237
02/15/2021	7.875	136.3438	136.3906	-0.0781	2.172

Table 10: Notes and Bonds 03/20/2014

05/15/2021	3.125	105.3203	105.3672	-0.1094	2.306
05/15/2021	8.125	138.7969	138.8438	-0.0469	2.220
08/15/2021	2.125	098.2109	098.2578	-0.0703	2.383
08/15/2021	8.125	139.6172	139.6641	-0.0703	2.275
11/15/2021	2.000	096.8672	096.9141	-0.0703	2.444
11/15/2021	8.000	139.4609	139.5078	-0.0547	2.332
02/15/2022	2.000	096.4688	096.5156	-0.0703	2.488
05/15/2022	1.750	094.1406	094.1875	-0.0313	2.544
08/15/2022	1.625	092.6875	092.7344	-0.0313	2.592
08/15/2022	7.250	136.2500	136.2969	0.0313	2.446
11/15/2022	1.625	092.2266	092.2344	0.0313	2.634
11/15/2022	7.625	139.8203	139.8672	0.0469	2.477
02/15/2023	2.000	094.7422	094.8047	0.0078	2.659
02/15/2023	7.125	136.2813	136.3281	0.0391	2.541
05/15/2023	1.750	092.2500	092.3125	-0.0078	2.704
08/15/2023	2.500	098.1719	098.2344	-0.0313	2.714
08/15/2023	6.250	130.2578	130.3047	0.1953	2.597
11/15/2023	2.750	100.0156	100.0781	-0.0156	2.740
02/15/2024	2.750	099.7422	099.8047	-0.0625	2.773
11/15/2024	7.500	143.7656	143.7891	0.1172	2.733
02/15/2025	7.625	145.4844	145.5625	0.1172	2.758
08/15/2025	6.875	139.0547	139.1328	0.1484	2.835
02/15/2026	6.000	130.8359	130.9141	0.1641	2.909
08/15/2026	6.750	139.3906	139.4688	0.2031	2.932
11/15/2026	6.500	137.0313	137.1250	0.2109	2.960
02/15/2027	6.625	138.6797	138.7578	0.2031	2.984
08/15/2027	6.375	136.4609	136.5391	0.2656	3.036
11/15/2027	6.125	133.8672	133.9453	0.2578	3.063
08/15/2028	5.500	127.0781	127.1563	0.2422	3.142
11/15/2028	5.250	124.2266	124.3047	0.2578	3.164

Table 11: Notes and Bonds 03/20/2014

02/15/2029	5.250	124.2266	124.3047	0.2891	3.188
08/15/2029	6.125	135.3672	135.4453	0.2422	3.194
05/15/2030	6.250	137.7656	137.8438	0.2813	3.225
02/15/2031	5.375	126.8438	126.9219	0.2656	3.286
02/15/2036	4.500	115.9141	115.9922	0.2969	3.453
02/15/2037	4.750	119.9063	119.9844	0.2266	3.477
05/15/2037	5.000	123.9219	124.0000	0.2266	3.481
02/15/2038	4.375	113.6328	113.6797	0.1953	3.523
05/15/2038	4.500	115.7109	115.7891	0.1094	3.523
02/15/2039	3.500	098.7969	098.8125	0.2109	3.572
05/15/2039	4.250	111.5313	111.6094	0.1797	3.548
08/15/2039	4.500	115.9609	116.0078	0.3359	3.539
11/15/2039	4.375	113.7344	113.8125	0.3047	3.550
02/15/2040	4.625	118.1484	118.2266	0.2344	3.544
05/15/2040	4.375	113.8047	113.8828	0.2969	3.555
08/15/2040	3.875	104.9375	105.0156	0.2813	3.580
11/15/2040	4.250	111.5391	111.6172	0.2578	3.570
02/15/2041	4.750	120.5547	120.6016	0.2891	3.554
05/15/2041	4.375	113.7891	113.8672	0.2813	3.573
08/15/2041	3.750	102.4844	102.5625	0.2656	3.602
11/15/2041	3.125	091.0234	091.1016	0.1875	3.638
02/15/2042	3.125	090.9141	090.9609	0.2188	3.644
05/15/2042	3.000	088.4922	088.5234	0.1719	3.656
08/15/2042	2.750	083.7734	083.7891	0.1797	3.674
11/15/2042	2.750	083.5703	083.6016	0.1563	3.681
02/15/2043	3.125	090.3438	090.4063	0.1406	3.666
05/15/2043	2.875	085.6016	085.6641	0.1172	3.681
08/15/2043	3.625	099.4063	099.4688	0.1406	3.654
11/15/2043	3.750	101.6875	101.7500	0.1484	3.653
02/15/2044	1.375	099.5664	099.9219	-0.6211	1.378

Table 12: Notes and Bonds 03/20/2014

02/15/2044	3.625	099.3438	099.4063	0.1719	3.658

5.2 All Bills with Reduced Number of bonds following [6, (32)]

Table 13: T-Bills 03/20/2014

T-Bill	01	Maturity	03/27/2014	YR =	0.019178	AskYield	0.000410	Price	99.999214
T-Bill	02	Maturity	04/03/2014	YR =	0.038356	AskYield	0.000300	Price	99.998849
T-Bill	03	Maturity	04/10/2014	YR =	0.057534	AskYield	0.000460	Price	99.997353
T-Bill	04	Maturity	04/17/2014	YR =	0.076712	AskYield	0.000560	Price	99.995704
T-Bill	05	Maturity	04/24/2014	YR =	0.095890	AskYield	0.000560	Price	99.994630
T-Bill	06	Maturity	05/01/2014	YR =	0.115068	AskYield	0.000460	Price	99.994707
T-Bill	07	Maturity	05/08/2014	YR =	0.134247	AskYield	0.000460	Price	99.993825
T-Bill	08	Maturity	05/15/2014	YR =	0.153425	AskYield	0.000350	Price	99.994630
T-Bill	09	Maturity	05/22/2014	YR =	0.172603	AskYield	0.000350	Price	99.993959
T-Bill	10	Maturity	05/29/2014	YR =	0.191781	AskYield	0.000410	Price	99.992137
T-Bill	11	Maturity	06/05/2014	YR =	0.210959	AskYield	0.000460	Price	99.990296
T-Bill	12	Maturity	06/12/2014	YR =	0.230137	AskYield	0.000460	Price	99.989414
T-Bill	13	Maturity	06/19/2014	YR =	0.249315	AskYield	0.000560	Price	99.986039
T-Bill	14	Maturity	06/26/2014	YR =	0.268493	AskYield	0.000460	Price	99.987650
T-Bill	15	Maturity	07/03/2014	YR =	0.287671	AskYield	0.000460	Price	99.986768
T-Bill	16	Maturity	07/10/2014	YR =	0.306849	AskYield	0.000460	Price	99.985886
T-Bill	17	Maturity	07/17/2014	YR =	0.326027	AskYield	0.000560	Price	99.981744
T-Bill	18	Maturity	07/24/2014	YR =	0.345205	AskYield	0.000560	Price	99.980670
T-Bill	19	Maturity	07/31/2014	YR =	0.364384	AskYield	0.000350	Price	99.987247
T-Bill	20	Maturity	08/07/2014	YR =	0.383562	AskYield	0.000560	Price	99.978523
T-Bill	21	Maturity	08/14/2014	YR =	0.402740	AskYield	0.000560	Price	99.977449
T-Bill	22	Maturity	08/21/2014	YR =	0.421918	AskYield	0.000610	Price	99.974266
T-Bill	23	Maturity	08/28/2014	YR =	0.441096	AskYield	0.000560	Price	99.975302
T-Bill	24	Maturity	09/04/2014	YR =	0.460274	AskYield	0.000660	Price	99.969627
T-Bill	25	Maturity	09/11/2014	YR =	0.479452	AskYield	0.000710	Price	99.965965
T-Bill	26	Maturity	09/18/2014	YR =	0.498630	AskYield	0.000760	Price	99.962111
T-Bill	27	Maturity	10/16/2014	YR =	0.575342	AskYield	0.000710	Price	99.959159
T-Bill	28	Maturity	11/13/2014	YR =	0.652055	AskYield	0.000910	Price	99.940681
T-Bill	29	Maturity	12/11/2014	YR =	0.728767	AskYield	0.000960	Price	99.930063
T-Bill	30	Maturity	01/08/2015	YR =	0.805479	AskYield	0.001010	Price	99.918680

Table 14: T-Bills 03/20/2014

T-Bill	31	Maturity	02/05/2015	YR =	0.882192	AskYield	0.001120	Price	99.901243
T-Bill	32	Maturity	03/05/2015	YR =	0.958904	AskYield	0.001270	Price	99.878293

Table 15: Bonds and Notes 03/20/2014

Maturity	Coupon	Bid	Asked	Chg	AskYield
03/31/2015	0.250	100.0938	100.1016	unch.	0.151
04/30/2015	0.125	099.9297	099.9766	-0.0078	0.146
05/31/2015	0.250	100.0547	100.1016	-0.0078	0.165
08/15/2015	0.250	100.0078	100.0547	0.0313	0.211
10/15/2015	0.250	099.9141	099.9375	0.0234	0.290
11/15/2015	0.375	100.0625	100.0859	0.0156	0.323
11/30/2015	0.250	099.8438	099.8750	unch.	0.324
12/31/2015	0.250	099.7891	099.8281	0.0156	0.347
02/15/2016	0.375	099.9219	099.9297	0.0234	0.412
03/31/2016	2.375	103.8672	103.8828	-0.0078	0.449
07/15/2016	0.625	100.0391	100.0781	unch.	0.591
08/31/2016	1.000	100.8281	100.8594	-0.0078	0.645
11/15/2016	0.625	099.6641	099.7031	-0.0078	0.738
12/15/2016	0.625	099.5469	099.5859	-0.0469	0.778
02/15/2017	0.625	099.3047	099.3438	-0.0391	0.854
03/15/2017	0.750	099.5547	099.5938	-0.0547	0.888
03/31/2017	1.000	100.2109	100.2266	-0.0469	0.924
03/31/2017	3.250	106.9766	107.0156	-0.0469	0.896
05/31/2017	0.625	098.7969	098.8047	-0.0469	1.006
08/31/2017	1.875	102.5156	102.5469	-0.0469	1.119
09/30/2017	0.625	098.0859	098.1016	-0.0703	1.176
09/30/2017	1.875	102.4609	102.4922	-0.0391	1.152
10/31/2017	0.750	098.3594	098.3750	-0.0391	1.211
12/31/2017	0.750	098.0234	098.0313	-0.0391	1.285
02/28/2018	0.750	097.6641	097.6953	-0.0469	1.352
03/31/2018	2.875	105.8984	105.9297	-0.0078	1.357
07/31/2018	2.250	103.1641	103.2266	-0.0625	1.484
09/30/2018	1.375	099.0781	099.1250	-0.0703	1.576
02/28/2019	1.500	098.9844	099.0313	-0.0703	1.705
03/31/2019	1.500	098.8125	098.8203	-0.0781	1.746

Table 16: Notes and Bonds 03/20/2014

05/31/2019	1.125	096.5703	096.6172	-0.0625	1.810
09/30/2019	1.000	095.1563	095.1719	-0.0625	1.925
10/31/2019	1.250	096.3047	096.3516	-0.0859	1.939
02/29/2020	1.250	095.5078	095.5547	-0.0781	2.048
05/31/2020	1.375	095.6875	095.7344	-0.1016	2.113
08/31/2020	2.125	099.6719	099.7188	-0.1094	2.172
10/31/2020	1.750	097.1016	097.1484	-0.0547	2.216
12/31/2020	2.375	100.7578	100.8047	-0.0859	2.246
01/31/2021	2.125	099.0156	099.0625	-0.0625	2.273
02/15/2022	2.000	096.4688	096.5156	-0.0703	2.488
08/15/2022	1.625	092.6875	092.7344	-0.0313	2.592
11/15/2022	1.625	092.2266	092.2344	0.0313	2.634
11/15/2028	5.250	124.2266	124.3047	0.2578	3.164
08/15/2029	6.125	135.3672	135.4453	0.2422	3.194
11/15/2039	4.375	113.7344	113.8125	0.3047	3.550
05/15/2040	4.375	113.8047	113.8828	0.2969	3.555
08/15/2041	3.750	102.4844	102.5625	0.2656	3.602
05/15/2042	3.000	088.4922	088.5234	0.1719	3.656
08/15/2042	2.750	083.7734	083.7891	0.1797	3.674
02/15/2043	3.125	090.3438	090.4063	0.1406	3.666

References

[1] F. L. Chernousko. *The Estimating of State for Dynamic Systems*. Nauka, Moscow, Russia, 1989. in Russian.

[2] R. Gabasov, F. M. Kirillova, and O. I. Kostyukova. *Constructive Methods of Optimization. Part 2. Control Problems*. University Press Belarus, Minsk, Belarus, 1984.

[3] R. Gabasov, F. M. Kirillova, and S. Prischepova. *Optimal Feedback Control*. Number 207 in Lecture Notes in Economica and Information Systems. Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1995.

[4] M. I. Gusev and S. A. Romanov. On stability of guaranted estimation problems: error bounds for information domains and experimental design. In M. A. Goberna and M. A. López, editors, *Semi-infinite Programming Recent Advances*, Nonconvex Optimization and Its Applications Volume 57, pages 255–326, 101 Norwell Drive, MA 02061, U. S. A, 2001. Kluwer Academic Publishers.

[5] K. O. Kortanek and V. G. Medvedev. *Building and Using Dynamic Interest Rate Models*. Wiley Finance. John Wiley & Sons Ltd, Baffins Lane, Chichester, West Sussex PO19 1UD, England, 2001.

[6] K. O. Kortanek and V. G. Medvedev. An optimization model for extracting forward interest rates from a dynamical system under financial uncertainty. *Dynamics of Continuous, Discrete & Impulsive Systems: Series B Applications & Algorithms*, 17:1–21, 2010. Special Issue on Optimization and Dynamics in Finance.

[7] Ken O. Kortanek. Nonstochastic uncertainty approaches to prediction in finance. In Heping Pan, Didier Sornette, and Kenneth Kortanek, editors, *Proceedings of the First International Workshop on Intelligane Finance(IWIF1)*, pages 144–171. Society of Information Technology and Mathematical Sciences, University of Ballarat, Australia, December 2004.

[8] N. N. Krasovsky. *Theory of Control with Movement*. Nauka, Moscow, Russia, 1976. in Russian.

[9] A. B. Kurzanski. *Control and Observation in Indefiniteness Conditions*. Nauka, Moscow, Russia, 1977. in Russian.

[10] A. B. Kurzhanski and V. M. Veliov, editors. *Modeling Techniques for Uncertain Systems*, Progress in Systems and Control Theory Volume 18, A-2361 Laxenburgh/Austria, 1994. Birkhäuser;International Institute for Applied Systems Analysis.

[11] V. Křivan and G. Colombo. A non-stochastic approach for modeling uncertainty in population dynamics. *Bulletin of Mathematical Biology*, 60:721–751, 1998.

[12] A. A. Martynuk and Yu. A. Martyyukn-Chernienko. *Uncertain Dynamical Systems. Stability and Motion Control*. CRC Press, Boca Raton, FL, 2012. Reviewed by Martin Bohner, Missouri University of Science and Technology, SIAM Review March 2014, Vol. 56, No. 1 pages 191-193.

[13] V. G. Medvedev. Algorithm for solving the optimal observation problem with non-convex a priori distribution of initial state. *Automatic Remote Control*, 10:121–130, 1993.

[14] V. G. Medvedev. Optimal observations of initial state and input disturbances for dynamic systems. *SAMS*, 14:275–288, 1994.

[15] V. G. Medvedev. Positional algorithm for optimal observations of linear dynamical systems. *SAMS*, 16:93–111, 1994.

[16] V. G. Medvedev. The method of construction of approximate solutions of linear optimal control problems with state constraints. Technical report, Department of Optimal Control Methods, Byelorussian State University, F. Skorina 4, 220050 Minsk, Belarus, 1995. appeared in Proc. XII International Conference on Proclaw, Poland, September 12-15, 1995.

[17] Lars E. O. Svensson. Estimating forward interest rates with the extended Nelson & Siegel method. *Sveriges Riksbank Quarterly Review*, 3:13–26, 1995.

[18] R. Tichatschke, A. Kaplan, T. Voetmann, and M. Böhm. Numerical treatment of an asset price model with non-stochastic uncertainty. *TOP Sociedad de Estadistíca e Investigación Operativa*, 10:1–50, 2002.