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Introduction to the Mathematics of Fixed Income Pricing

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Introduction

Powerfully built techniques for handling the dynamics and calculus of stochastic variables such as interest rates have been developed over the last few decades. In this section we introduce the fundamentals of mathematical finance with respect to fixed income pricing. An extended and through discussion of the content of this section can be found in Choudhry (2004).

To begin we need to state the following sets of assumptions, generally adopted from Merton's[1] pricing method:

- There are no transaction cost or taxes
- There exists an exchange market for borrowing and lending at the same rate of interest (no bid-offer spread)
- The term structure is "flat" and known with certainty
- There is a rational and competitive market
- Market participants prefer to increase wealth
- There are no arbitrage opportunities.

The main prerequisite of mathematical finance that is imperative in understanding fixed income are risk neutral valuation and arbitrage pricing theory. In this introduction we will establish the probabilistic setting in which these concepts are formulated.

As stated in Musiela and Rutkowski (1998), an economy is a family of filtered space $\{(\Omega, I, \mu) : \mu \in P\}$[2], where the filtration satisfies the usual conditions[3], and ***P*** is a collection of mutually equivalent probability measures on the measurable space.[4] We model the subjective market uncertainty of each investor by associating to each investor a probability measure from ***P***. Investors with more risky tolerance will be represented by probability measures that weight unfavourable events relatively lower, whereas conservatives investors are characterized by probability measures that weight unfavourable relatively higher. Moreover, it is assumed that investment information is revealed to each investor simultaneously as events in the filtration.

Since the measures in ***P*** are mutually equivalent, the investors agree on the events that have and have not occurred. We refer the reader to Neftci (2000) for an excellent example of this. It is convenient to further assume that investors initially have no other information, that is, the filtration is trivial with respect to each probability measure in ***P***. This assumption asserts that the initial information available to investors is objective.

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[1] Robert C. Merton "Continuous-Time Finance" 1998

[2] To forecast a random variable, one utilizes some information denoted by the symbol I_t. See more of this Neftci (2000) pp. 97

[3] See Karatzas and Shreve (1991) section 2.7 and Korn and Korn (2000) pp. 18.

[4] For a definition of measurable space and how to construct this see Jeffrey S. Rosenthal (2000) section 2.

The foundation of a working knowledge of fixed income finance rests on an understanding of the inherent relationship between the various interest rates and bonds. Consider the economy $\{(\Omega, I, \mu): \mu \in P\}$ on the interval $[0,T]$ and a Markov process [5] X_t with $I \equiv \sigma(X_{:0\le s\le t})$. Implicit in this statement is the assumption that the state variable[6] probability $P \equiv P^X$ associated with X_t belongs to ***P*** for some fixed elements X of the state space X_t.

Setting the scene further, a zero coupon or discount bond of maturity *T* is a security that pays the holder one unit of currency at time *T*. The prices of government and corporate discount bonds at time $t \le T$ are denoted $B(t,T)$ and $\widetilde{B}(t,T)$ respectively. The local expectation hypothesis (L-EH) relates the discount bond to the instantaneous interest rate, or the spot rate for borrowing of the loan over the time interval $[t, t+dt]$.

Denote the riskless spot rate by $r_t = r(X_t)$ and assume that it is a non-negative, adapted process with almost all sample paths integrable on the $[0,T]$ with respect to the Lebesgue measure.

The L-EH asserts that

$$B(t,T) = E_P(\exp(-\int_t^T r(X_s)ds \,|\, I_t)$$

As defined in Musiela and Rutkowski (1998), the economic interpretation of this hypothesis is that "… the current bond price equals the expected value … of the bond price in the next (infinitesimal) period, discounted at the current short-term rate".[7]

This statement is better understood in a discrete time setting. In fact using a left sum approximation to the integral with partition $\{t_i\}_{t=0}^{n}$ of $[0,T]$ yields

$$\begin{aligned} B(0,T) &= E_P(\exp(-\sum_{i=L}^{n} r(X_{t_{i-1}})\Delta t_i)) \\ &= E_P(\exp(-r(X_{t_o})\Delta t_L)\exp(-\sum_{i-2}^{n} r(X_{t_{i-1}})\Delta t_t)) \\ &= (\exp(-r(X)t_L)E_P(E_P(\exp(-\sum_{i-2}^{n} r(X_{t_{i-1}})\Delta t_t \,|\, I_{t_1})) \\ &= (\exp(-r(X_{t_o})t_L)E_P^X(B(t,T)) \end{aligned}$$

[5] See Choudhry (2004) pp. 185 for a lucid definition of the Markov process.
[6] See Choudhry (2004), pp. 144.
[7] See ibid, pp. 283.

Under the assumption of no arbitrage, it can be shown that above equation holds under the risk neutral measure. [8] Naturally as similar relationship holds between the risky bond and the risky spot rate.

Bond pricing

The process B_t is referred to as an accumulation factor or savings account.[9] B_t represents the price of a riskless security that continuously compounds at the spot rate. More precisely it is the amount of cash at time *t* that accumulates by investing $1 initially, and continually rolling over a bond with an infinitesimal time to maturity. See Musiela and Rutkowski (1998) page 268 for more detail on this.

Therefore an adapted process B_t of finite variation with continuous sample path is given by

$$B_t = \exp(\int_0^t r(X_s)ds$$

When security S_t is divided by the saving account the resultant process is the price process of the security discounted at the riskless rate. [10]

We consider next a coupon-bearing bond, with fixed coupon payments $c_1,...,c_n$ at predetermined times $T_1,...,T_n$ with $T_n = T$.

The price of the coupon bond is simply the present value of the sum of these cash flows. Denoting the price of a riskless coupon bond at time by $cB(t,T)$, we have

$$B_c(t,T) = \sum_{i=1}^{n} c_i B(t,T)$$

A similar relationship holds for the risky coupon bond.

However the coupons are typically structured by setting $c_i = c$ for i $= 1,...,n-1$ and $c_n = N + c$, where *N* is the principal or face value, and *c* is a fixed amount that is generally quoted as a percentage of *N* called the coupon rate. A problem that arises in comparing coupon bonds is that uncertainty about the rate at which the coupons will be reinvested causes uncertainty in the total return of the coupon bond. Hence, coupon bonds of different coupon rates and payment dates are not directly comparable. The standard way to overcome this problem is to extend the notion of a yield to maturity to coupon bearing bonds.

[8] We refer to the interested reader to Ingersoll (1987).
[9] See Pliska (1997) chapter 1.
[10] In other words, the bank account is the Numeraire.

Yield to Maturity (YTM)
In Musiela and Rutkowski (1998) the continuously compounded riskless yield to maturity $Y_c(t) = Y_c(t; c_1, ..., c_n . T_1, ..., T_n)$ is derived as the unique exposition to the equation

$$B_c(t) = \sum_{T_i > t} c_i e^{-Y_c(t)(T_i - t)}$$

and stands for the total return on the coupon bond under the assumption that each of the coupon payments occurring after *t* is reinvested at the rate $Y_c(t)$. The risky yield to maturity is defined in a similar fashion.

Expectation Hypothesis
There are number of excellent textbooks that the reader is encouraged to read which provides the necessary background, in particular Ingersoll (1987) and Choudhry (2004).

The yield to maturity expectation hypothesis (YTM-EH) relates the riskless YTM and the riskless spot rate. Musiela and Rutkowski (1998) state that this hypothesis as the assertion that

> "… the [continuously compounded] yield from holding any [discount] bond is equal to the [continuously compounded yield expected from rolling over a series of single period [discount] bonds".

To gain a better understanding of this statement, we first observe that the YTM of a discrete time setting with the partition $\{t_i\}_{i=0}^{n}$ of $[t, T]$, we have that the yield of a discount bond $B(t_{i-1}, t_i)$ is given by

$$Y(t_{i-1}, t_i) = r(X_{t_{i-1}})$$

from which we deduce that the bond price is given by

$$B(t_{i-1}, t_i) = \exp(-r(X_{t_{i-1}})\Delta t_i \, .$$

Since the YTM-EH asserts that the yield of $B(t, T)$ is equal to the yield expected from rolling over a series of discount bonds $B(t_{i-1}, t_i)$, it follows that

$$Y(t,T) = \frac{1}{T-t}\ln(B(t,T)) = -\frac{1}{T-t} E_P\left[\ln\left(\prod_{i-L}^{n} B(t_{i-L}, t_i)\right) \middle| I_t\right]$$
$$= \frac{1}{T-t} E_P\left[\sum_{i-L}^{n} r(X_{t_{i-1}})\Delta t_t \middle| I_t\right].$$

Taking the limit, as the mesh of the partition tends to zero; we obtain the continuously time discount bond price and YTM under the YTM-EH:

$$B(t,T) = \exp\left[-E_P\left(\int_t^T r(X_s)ds\right)\middle\| I_t\right]$$

$$Y(t,T) = \frac{1}{T-t}\left[E_P\left(\int_t^T r(X_s)ds\right)\middle\| I_t\right]$$

The last interest rate that we will consider is the instantaneous forward interest rate, or forward rate for borrowing or lending over the time interval $[s, s+ds]$ as seen from time $[t \le s]$. This will be denoted by $f(t,s)$ in the riskless case and $\widetilde{f}(t,s)$ in the risky case.

If the dynamics of the process $\{f(t,s)\}_{t\le s\le T}$ are specified, then the price of the discount bond is defined by

$$B(t,T) = \exp(-\int_t^T f(s,t)ds$$

Alternatively, if the dynamics of the discount bond are known, then we have

$$f(t,T) = -\frac{\partial}{\partial T}\ln B(t,T),$$

provided that this derivation exists!

Therefore the YTM-EH asserts that the forward rate is an unbiased estimate of the spot rate under the state variable probability measure ***P***. See Choudhry (2004) chapter 2, equation (2.18). For the relationship between the spot and forward rate we refer the reader to read further in chapter 3 of Choudhry (2004).

Review of Arbitrage Pricing Theory[11]

The methodology presented in this review can be found in Musiela and Rutkowski (1998)[12]. Consider the economy $\{(\Omega, I, \mu) : \mu \in P\}$ on the interval $[0,T]$. A trading strategy or portfolio ϕ_t is a vector of locally bounded adapted processes of tradable asset holdings. Moreover, it is assumed that every sample path is right continuous with left limits[13]. A trading strategy ϕ_t is called self-financing if the wealth process $W_t(\phi)$ of the trading strategy neither receives nor pays out cash flows external to the assets that comprise the strategy.

[11] See Bjork (1997) for detailed discussion.
[12] See pages 72, 82, 188 and 231-232.
[13] Usually denoted RCLL and also known as cadlag! See Karatzas and Shreve (1991) page 4.

More precisely, let ϕ^i denote the holding of asset S^i. Then, a self-financing trading strategy $\phi = (\phi^L, ..., \phi^n)$ is defined by asserting that $W_t(\phi) = \sum_{t=L}^{n} \phi_t^L S_t^i$ satisfies

$$dW_t(\phi) = \phi_t^L dS_t^L + ... + \phi_t^L dS_t^n$$

A strategy $\phi \in \Phi_T$ is called an arbitrage opportunity if the wealth process $W(\phi)$ satisfies for some (consequently for all) $P \in \boldsymbol{P}$, all the following conditions

$$\begin{aligned} &W_0(\phi) = 0 && \text{(Zero investment)} \\ &\mathrm{P}\{\mathrm{W_T}(\phi) \geq 0\} = 1 && \text{(Zero risk)} \\ &\mathrm{P}\{\mathrm{W_T}(\phi) > 0\} > 0 && \text{(Possible gain).} \end{aligned}$$

Thus, taking an advantage the arbitrage opportunity it is possible to create limitless wealth without risk. Under the assumption that arbitrage portfolios do not exist, it has been shown that there exists a risk-neutral or martingale measure Q in our economy under which the discounted asset process $Z \equiv B_t^{-1} S_t$ follows a martingale. This result is known as the Fundamental Theorem of Asset Pricing. Musiela and Rutkowski (1998) define this theorem as “a result, which establishes the equivalence the absence of an arbitrage opportunity in the stochastic model of financial market, and the existence of a martingale measure”.

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