



ILLUSTRATING SPOT AND FORWARD INTEREST RATES
Learning Curve
August 2003

Practitioners in the bond markets need to determine the true interest rate for any period or term to maturity, for a number of applications. Such interest rates are known as *spot* or *zero-coupon* rates, because they would be the interest rates payable on a zero-coupon bond. However we do not need to have a market in zero-coupon bonds in order to calculate spot interest rates; we can calculate them from the yields observed on ordinary coupon bonds. We illustrate how to do so in this article.

Calculating Implied Spot and Forward Interest Rates

We describe how to obtain zero-coupon and forward interest rates from the yields available from coupon bonds, using a method known as *boot-strapping*. In a government bond market such as that for US Treasuries or gilts, the bonds are considered to be *credit-risk free* or *default-free*, because we assume the bonds will not default during their life. The rates from a government bond yield curve describe the risk-free rates of return available in the market *today*, however they also *imply* (risk-free) rates of return for *future time periods*. These implied future rates, known as *implied forward rates*, or simply *forward rates*, can be derived from a given spot yield curve using boot-strapping. This term reflects the fact that each calculated spot rate is used to determine the next period spot rate, in successive steps. We first show how to obtain the spot rates, and then the forward rates.

Table 1 shows an hypothetical benchmark yield curve for value as at 2 January 2000. The observed yields of the benchmark bonds that compose the curve are displayed in the last column. All rates are annualised and assume semi-annual compounding. The bonds all pay on the same coupon dates of 2 January and 2 July, and as the value date is a coupon date, there is no accrued interest.

Hypothetical government benchmark bond yields as at 2 January 2000					
Bond	Term to maturity (years)	Coupon	Maturity date	Price	Gross Redemption Yield
6-month	0.5	4%	02-Jul-00	100	4%
1-year	1	5%	02-Jan-01	100	5%
1.5 year	1.5	6%	02-Jul-01	100	6%
2-year	2	7%	02-Jan-02	100	7%

Table 1

The gross redemption yield or *yield-to-maturity* of a coupon bond describes the single rate that present-values each of its future cash flows to a given price. This yield measure suffers from a fundamental weakness in that each cash-flow is present-valued at the same rate, an unrealistic assumption in anything other than a flat yield curve environment. The bonds in Table 1 pay semi-annual coupons on 2 January and 2 July and have the same time period - six months - between 2 January 2000, their valuation date and 2 July 2000, their next coupon date. However since each issue carries a different yield, the next six-month coupon payment for each bond is present-valued at a different rate. In other words, the six-month bond present-values its six-month coupon payment at its 4% yield to maturity, the one-year at 5%, and so on.

Because each of these issues uses a different rate to present-value a cash flow occurring at the same future point in time, it is unclear which of the rates should be regarded as the true interest rate or benchmark rate for the six-month period from 2 January 2000 to 2 July 2000. This problem is repeated for all other maturities. We require a set of true interest rates however, and so these must be derived from the redemption yields that we can observe from the benchmark bonds trading in the market. These rates we designate as rs_i , where rs_i is the *implied spot rate* or *zero-coupon rate* for the term beginning on 2 January 2000 and ending at the end of period i .

We begin calculating implied spot rates by noting that the six-month bond contains only one future cash flow, the final coupon payment and the redemption payment on maturity. This means that it is in effect trading as a zero-coupon bond. Since this cash flow's present value, future value and maturity term are known, the unique interest rate that relates these quantities can be solved using the compound interest equation (1) below.

$$FV = PV \times \left(1 + \frac{rs_i}{m}\right)^{(nm)} \quad (1)$$

$$rs_i = m \times \left(\sqrt[nm]{\frac{FV}{PV}} - 1 \right)$$

where

- FV is the future value
- PV is the present value
- rs_i is the implied i -period spot rate
- m is the number of interest periods per year
- n is the number of years in the term

The first rate to be solved is referred to as the implied six-month spot rate and is the true interest rate for the six-month term beginning on 2 January and ending on 2 July 2000.

Equation (1) relates a cash flow's present value and future value in terms of an associated interest rate, compounding convention and time period. Of course if we re-arrange it, we may use it to solve for an implied spot rate. For the six-month bond the final cash flow on maturity is £102, comprised of the £2 coupon payment and the par redemption amount. So we have for the first term, $i = 1$, $FV = £102$, $PV = £100$, $n = 0.5$ years and $m = 2$. This allows us to calculate the spot rate as follows :

$$\begin{aligned}
 rs_i &= m \times \left(\sqrt[m]{FV/PV} - 1 \right) \\
 rs_1 &= 2 \times \left(\sqrt[2]{£102/£100} - 1 \right) \\
 rs_1 &= 0.04000 \\
 rs_1 &= 4.000\%
 \end{aligned}
 \tag{2}$$

Thus the implied six-month spot rate or zero-coupon rate is equal to 4%.¹ We now need to determine the implied one-year spot rate for the term from 2 January 2000 to 2 January 2001. We note that the one-year issue has a 5% coupon and contains two future cash flows : a £2.50 six-month coupon payment on 2 July 2000 and a £102.50 one-year coupon and principal payment on 2 January 2001. Since the first cash flow occurs on 2 July - six months from now - it must be present-valued at the 4% six-month spot rate established above. Once this present value is determined, it may be subtracted from the £100 total present value of the one-year issue to obtain the present value of the one-year coupon and cash flow. Again we then have a single cash flow with a known present value, future value and term. The rate that equates these quantities is the implied one-year spot rate. From equation (1) the present value of the six-month £2.50 coupon payment of the one-year benchmark bond, discounted at the implied six-month spot rate, is :

$$\begin{aligned}
 PV_{6\text{-mo cash flow, 1-yr bond}} &= £2.50/(1 + 0.04/2)^{(0.5 \times 2)} \\
 &= £2.45098
 \end{aligned}$$

The present value of the one-year £102.50 coupon and principal payment is found by subtracting the present value of the six-month cash flow, determined above, from the total present value (current price) of the issue :

$$\begin{aligned}
 PV_{1\text{-yr cash flow, 1-yr bond}} &= £100 - £2.45098 \\
 &= £97.54902
 \end{aligned}$$

The implied one-year spot rate is then determined by using the £97.54902 present value of the one-year cash flow determined above :

$$\begin{aligned}
 rs_2 &= 2 \times \left(\sqrt[2]{£102.50/£97.54902} - 1 \right) \\
 &= 0.0501256 \\
 &= 5.01256\%
 \end{aligned}$$

¹ Of course intuitively we could have concluded that the six-month spot rate was 4 per cent, without the need to apply the arithmetic, as we had already assumed that the six-month bond was a quasi-zero-coupon bond.

The implied 1.5 year spot rate is solved in the same way:

$$\begin{aligned} \text{PV}_{6\text{-mo cash flow, 1.5-yr bond}} &= \text{£}3.00 / (1 + 0.04 / 2)^{(0.5 \times 2)} \\ &= \text{£}2.94118 \end{aligned}$$

$$\begin{aligned} \text{PV}_{1\text{-yr cash flow, 1.5-yr bond}} &= \text{£}3.00 / (1 + 0.0501256 / 2)^{(1 \times 2)} \\ &= \text{£}2.85509 \end{aligned}$$

$$\begin{aligned} \text{PV}_{1.5\text{-yr cash flow, 1.5-yr bond}} &= \text{£}100 - \text{£}2.94118 - \text{£}2.85509 \\ &= \text{£}94.20373 \end{aligned}$$

$$\begin{aligned} r_{s_3} &= 2 \times \left(\sqrt[1.5 \times 2]{\text{£}103 / \text{£}94.20373} - 1 \right) \\ &= 0.0604071 \\ &= 6.04071\% \end{aligned}$$

Extending the same process for the two-year bond, we calculate the implied two-year spot rate r_{s_4} to be 7.0906 per cent.

The interest rates r_{s_1} , r_{s_2} , r_{s_3} and r_{s_4} describe the true zero-coupon or spot rates for the 6-month, 1-year, 1.5-year and 2-year terms that begin on 2 January 2000 and end on 2 July 2000, 2 January 2001, 2 July 2001 and 2 January 2002 respectively. They are also called implied spot rates because they have been calculated from redemption yields observed in the market from the benchmark government bonds that were listed in Table 1.

Note that the one-, 1.5- and two-year implied spot rates are progressively greater than the corresponding redemption yields for these terms. This is an important result, and occurs whenever the yield curve is positively sloped. The reason for this is that the present values of a bond's shorter-dated cash flows are discounted at rates that are lower than the redemption yield; this generates higher present values that, when subtracted from the current price of the bond, produce a lower present value for the final cash flow. This lower present value implies a spot rate that is greater than the issue's yield. In an inverted yield curve environment we observe the opposite result, that is implied rates that lie below the corresponding redemption yields. If the redemption yield curve is flat, the implied spot rates will be equal to the corresponding redemption yields.

Once we have calculated the spot or zero-coupon rates for the six-month, one-year, 1.5-year and two-year terms, we can determine the rate of return that is implied by the yield curve for the sequence of six-month periods beginning on 2 January 2000, 2 July 2000, 2 January 2001 and 2 January 2001. These period rates are referred to as *implied forward rates* or *forward-forward rates* and we denote these as r_{f_i} , where r_{f_i} is the implied six-month forward interest rate today for the i th period.

Since the implied six-month zero-coupon rate (spot rate) describes the return for a term that coincides precisely with the first of the series of six-month periods, this rate describes the risk-free rate of return for the first six-month period. It is therefore equal to the first period spot rate. Thus we have $rf_1 = rs_1 = 4.0$ per cent, where rf_1 is the risk-free forward rate for the first six-month period beginning at period 1. The risk-free rates for the second, third and fourth six-month periods, designated rf_2 , rf_3 and rf_4 respectively may be solved from the implied spot rates.

The benchmark rate for the second semi-annual period rf_2 is referred to as the one-period forward six-month rate, because it goes into effect one six-month period from now (“one-period forward”) and remains in effect for six months (“six-month rate”). It is therefore the six-month rate in six months time, and is also referred to as the 6-month forward-forward rate. This rate in conjunction with the rate from the first period rf_1 , must provide returns that match those generated by the implied one-year spot rate for the entire one-year term. In other words, one pound invested for six months from 2 January 2000 to 2 July 2000 at the first period’s benchmark rate of 4 per cent and then reinvested for another six months from 2 July 2000 to 2 January 2001 at the second period’s (as yet unknown) implied *forward* rate must enjoy the same returns as one pound invested for one year from 2 January 2000 to 2 January 2001 at the implied one-year *spot* rate. This reflects the law of no-arbitrage.

A moment’s thought will convince us that this must be so. If this were not the case, we might observe an interest rate environment in which the return over any given term would depend on whether an investment is made at the start period for the entire maturity term or over a succession of periods within the whole term and re-invested. If there were any discrepancies between the returns received from each approach, there would exist an unrealistic arbitrage opportunity, in which investments for a given term carrying a lower return might be sold short against the simultaneous purchase of investments for the same period carrying a higher return, thereby locking in a risk-free, cost-free profit. Therefore forward interest rates must be calculated so that they are *arbitrage-free*.

The existence of a no-arbitrage market of course makes it straightforward to calculate forward rates; we know that the return from an investment made over a period must equal the return made from investing in a shorter period and successively reinvesting to a matching term. If we know the return over the shorter period, we are left with only one unknown – the full-period forward rate, which is then easily calculated. In our example, having established the rate for the first six-month period, the rate for the second six-month period - the one-period forward six-month rate - is determined below.

The future value of £1 invested at rf_1 , the period 1 forward rate, at the end of the first six-month period is calculated as follows :

$$\begin{aligned} FV_1 &= \text{£}1 \times \left(1 + \frac{rf_1}{2}\right)^{(0.5 \times 2)} \\ &= \text{£}1 \times \left(1 + \frac{0.04}{2}\right)^1 \\ &= \text{£}1.02000 \end{aligned}$$

The future value of £1 at the end of the one-year term, invested at the implied benchmark one-year spot rate is determined as follows :

$$\begin{aligned} FV_2 &= \text{£}1 \times \left(1 + \frac{rs_2}{2}\right)^{(1 \times 2)} \\ &= \text{£}1 \times \left(1 + \frac{0.0501256}{2}\right)^2 \\ &= \text{£}1.050754 \end{aligned}$$

The implied benchmark one-period forward rate rf_2 is the rate that equates the value of FV_1 (£1.02) on 2 July 2000 to FV_2 (£1.050754) on 2 January 2001. From equation (1) we have :

$$\begin{aligned} rf_2 &= 2 \times \left(\sqrt[0.5 \times 2]{\frac{FV_2}{FV_1}} - 1 \right) \\ &= 2 \times \left(\frac{\text{£}1.050754}{\text{£}1.02} - 1 \right) \\ &= 0.060302 \\ &= 6.0302\% \end{aligned}$$

In other words £1 invested from 2 January to 2 July at 4.0 per cent (the implied forward rate for the first period) and then reinvested from 2 July to 2 January 2001 at 6.0302 per cent (the implied forward rate for the second period) would accumulate the same returns as £1 invested from 2 January 2000 to 2 January 2001 at 5.01256 per cent (the implied one-year spot rate).

The rate for the third six-month period - the two-period forward six-month interest rate – may be calculated in the same way:

$$FV_2 = \pounds 1.050754$$

$$\begin{aligned} FV_3 &= \pounds 1 \times (1 + rs_3 / 2)^{(1.5 \times 2)} \\ &= \pounds 1 \times (1 + 0.0604071 / 2)^3 \\ &= \pounds 1.093375 \end{aligned}$$

$$\begin{aligned} rf_3 &= 2 \times \left(\sqrt[0.5 \times 2]{\frac{FV_3}{FV_2}} - 1 \right) \\ &= 2 \times \left(\sqrt[1]{\frac{\pounds 1.093375}{\pounds 1.050754}} - 1 \right) \\ &= 0.081125 \\ &= 8.1125\% \end{aligned}$$

In the same way the three-period forward six-month rate rf_4 is calculated to be 10.27247 per cent.

The results of the implied spot (zero-coupon) and forward rate calculations along with the given yield curve are displayed in Table 2, and illustrated graphically in Figure 1. This methodology can be applied using a spreadsheet for actual market redemption yields, although in practice we will not have a set of bonds with exact and/or equal periods to maturity and coupons falling on the same date. In designing a spreadsheet spot rate calculator therefore, the coupon rate and maturity date is entered as standing data and usually interpolation is used when calculating the spot rates for bonds with uneven maturity dates.

Term to maturity (years)	Cash market yield	Implied spot rate	Implied one-period forward rate
0.5	4.00000%	4.00000%	4.00000%
1	5.00000%	5.01256%	6.03023%
1.5	6.00000%	6.04071%	8.11251%
2	7.00000%	7.09062%	10.27247%

Table 2

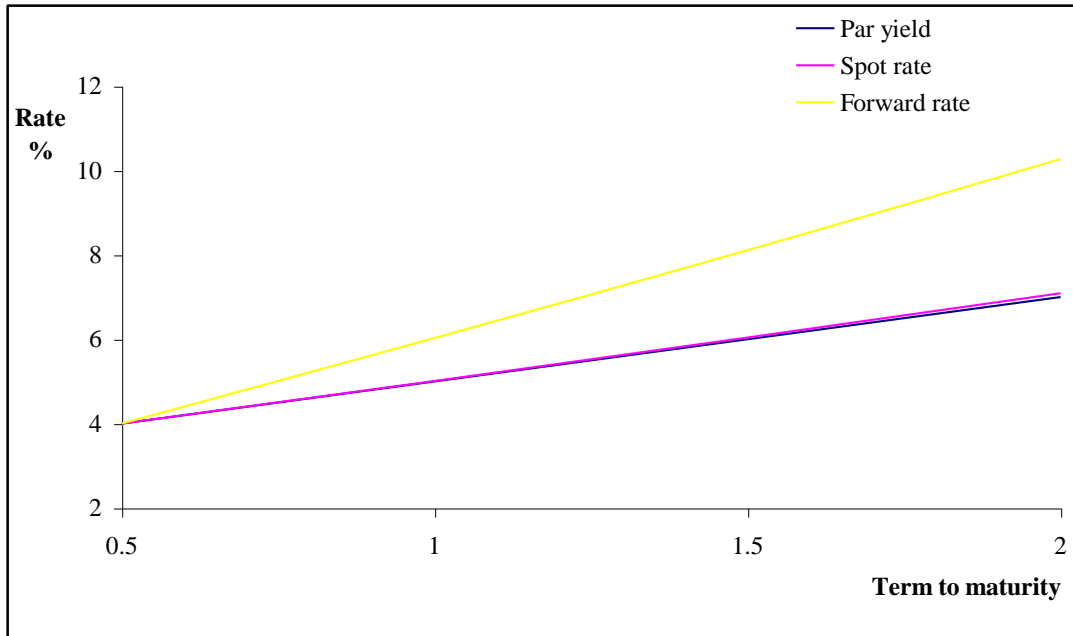


Figure 1